



The Open University

MST121
Using Mathematics

Chapter A0

Starting points

About MST121

This module, MST121 *Using Mathematics*, and the modules MU123 *Discovering mathematics* and MS221 *Exploring Mathematics* provide a flexible means of entry to university-level mathematics. See the address below for further details.

MST121 uses Mathcad (Parametric Technology Corporation) and other software to investigate mathematical and statistical concepts and as a tool in problem solving. This software is provided as part of MST121.

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Introduction to Block A

The title of this block, *Mathematics and modelling*, indicates one of the principal themes of MST121, namely ‘mathematical modelling’. This phrase describes the use of mathematics to represent, or ‘model’, objects and processes in the real world and, thereby, to try to obtain new information about these objects and processes.

Effective mathematical modelling requires several skills. First, you must acquire appropriate mathematical skills and develop facility at using them. Next, you need practice and experience at expressing real-world problems as mathematical problems, and an understanding of which mathematical techniques might be applied to solve these problems. Finally, you need appropriate computing skills and an appreciation of when the computer should be used in the solution of problems.

Chapter A0 provides a foundation for the development of your mathematical skills, and introduces the mathematical software that you will use in MST121. It takes you through the installation of the software package Mathcad, associated files and MST121 software, and provides revision material designed to indicate various ways in which Mathcad can support your mathematical work.

Chapter A1 deals with certain types of sequences that can be used to represent real-life processes as diverse as mortgage repayments and animal population behaviour. You will learn how to recognise such sequences and find formulas which make it possible to describe how the sequences behave.

Chapter A2 presents the use of algebraic techniques for representing lines and circles. These geometric objects are fundamental to the description of certain aspects of the real world, such as the construction of maps. Therefore, being able to perform calculations with lines and circles is important to our understanding of the real world.

Chapter A3 introduces the mathematical object known as a *function*, which can be described informally as ‘a process for converting inputs to outputs’. Many real-world problems can be expressed conveniently in terms of appropriate functions, so much mathematics is devoted to developing techniques for dealing with functions. In particular, this chapter shows how functions and their graphs can be used to solve problems involving areas and volumes.

Each of the chapters in this block includes teaching material which shows how Mathcad can be used with the topic being studied. This material is designed to help you understand when use of the computer is appropriate and to increase your range of computer skills in a systematic way.

‘Sequence’ is the mathematical name for an ordered list.

Study guide

There are four sections to this chapter, which are intended to be studied consecutively, and an appendix that you should read before moving on to Chapter A1. Each section should take between two and three hours of study. All sections require the use of this main text and the computer. Section 1 requires the Mathcad CD and the separate MST121/MS221 Software CD, while Sections 3 and 4 require the use of the calculator. For Sections 3 and 4 you may find it helpful to have the *Revision Pack* to hand.

The pattern of study for each session might be as follows.

Study session 1: Section 1 and Subsection 2.1.

Study session 2: The remainder of Section 2.

Study session 3: Section 3.

Study session 4: Section 4.

Solutions to activities that involve arithmetic or algebraic calculation start on page 41. Such activities based on the computer have on-screen solutions.

In this chapter you will need to use the following Mathcad files.

121A0-01 Worksheets expressions and text

121A0-02 Rearranging worksheets

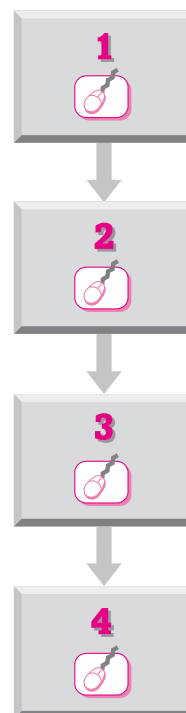
121A0-03 Calculating with Mathcad

121A0-04 Using variables

121A0-05 Rearranging algebraic expressions

Instructions for using these files are contained in Activities in this text.

All the Mathcad skills that you practise in this chapter will be revisited in later chapters.



The optional video band A(i), *At home with MST121*, is best viewed as early as possible in your studies. It shows two former MST121 students and a tutor describing their experiences of the module and giving some useful advice.

Introduction

This chapter has several important aims. It summarises the mathematical techniques in arithmetic and algebra on which MST121 builds, much of which should be familiar to you. It also introduces the mathematical software package Mathcad, which is an important resource for your study, and shows you how to use Mathcad to do arithmetic and algebra. As MST121 progresses, there will be many occasions where you can choose whether to carry out a particular piece of mathematics by hand, by calculator or by using Mathcad, so you need to begin to think here about the role of Mathcad in your work.

Section 1 explains how to install Mathcad, the files which are used with Mathcad and the MST121 software onto your computer. It is assumed here that you have access to a computer that satisfies the MST121 specification and that you have some experience of the *Microsoft Windows* environment. *Windows* is used in a straightforward way, and there are explicit instructions on what you have to do.

To install the software, you need this chapter and:

- ◇ the Mathcad CD;
- ◇ the MST121/MS221 Software CD;
- ◇ an individual Mathcad product code, to be found on your OU StudentHome web page;
- ◇ an internet link at the time of licensing Mathcad.

Licensing of Mathcad may be done separately from installation, but the installed software does not become active until it has been licensed.

Section 2 guides you through some first steps in using Mathcad. The two main uses of Mathcad in MST121 will be to perform routine calculations and to carry out investigations that would not otherwise be possible.

Section 3 looks at numbers and calculations. Numbers are classified into various categories, such as *integer* (whole number), *rational* (fraction) and *irrational* (cannot be expressed as a fraction). Then the basic rules of arithmetic, including powers, are reviewed in preparation for seeing how calculations are done in Mathcad.

Section 4 shifts attention to algebra. It briefly reviews the rules for rearranging algebraic expressions, and then the procedures for solving linear and quadratic equations. You will discover that Mathcad can perform algebraic operations, and you will see that it can deal with cases which would be difficult to do by hand.

It is advisable to access your product code before starting the installation process.

Both Sections 3 and 4 build on material in the *Revision Pack*, and you may like to consult this at times while studying this chapter.

1 *Installing the software*

1.1 *Installing and licensing Mathcad*

Switch on your computer and start running Windows.



Installing Mathcad (from the CD provided) and licensing it constitute the first of two stages in the installation of software for MST121.

Even if you already have a version of Mathcad on your computer, you should install the MST121/MS221 version, as described below. Versions of Mathcad may differ, and all references are to the MST121/MS221 version.

To ensure successful Mathcad installation, your computer needs to satisfy the following conditions:

- ◇ at least 550 MB of free space on your hard disk;
- ◇ Internet Explorer 6.0 or later;
- ◇ Adobe Acrobat Reader 5.0 or later;
- ◇ no applications running.

These items are offered during the Mathcad installation process.

Also, you must be logged into an account that has Administrator privileges. (You then need to log into the same account to run the software subsequently.)

If you have any worries about these conditions, or installation subsequently fails, then consult the MST121 website for information on whom to contact.

To install Mathcad, you will need the Mathcad CD and your individual Mathcad product code. The product code is to be found on your OU StudentHome web page, and it is worth seeking this before starting the installation.

Activity 1.1 *Installing and licensing Mathcad*

The desktop should be visible on your computer screen. If there are any applications running, then close them now.

Depending on which version of *Windows* you have, at some stages of the installation process a window may appear asking whether you wish to allow the program to run. If this happens, then you should click to allow it to do so.

- (a) Put the Mathcad CD into the DVD/CD drive. The installation process should start automatically, in which case the window shown in Figure 1.1 overleaf will appear.

If this does not happen, run `d:\setup` (where the first letter, `d` here, should be the letter corresponding to your DVD/CD drive). In recent versions of *Windows* open the **start** menu, type `d:\setup` in the *Start Search* box, then press the [Enter] key (which may alternatively be labelled [Return] or [↵]). In earlier versions of *Windows* open the **start** menu and select the **Run...** option. The 'Run' dialogue box will appear. In the **Open** box type `d:\setup`, then click on the **OK** button.

You might find it helpful to read through this activity before carrying out the various steps.

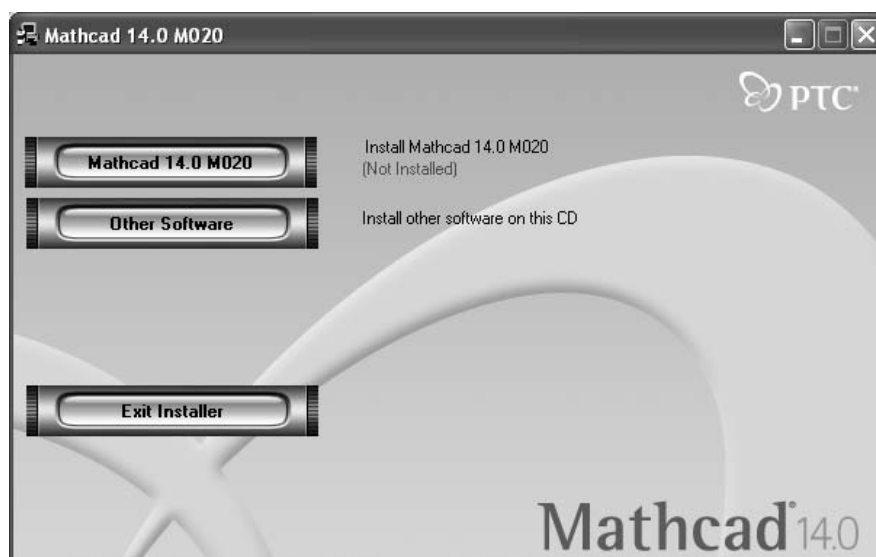


Figure 1.1 Mathcad installation launch window

The ‘other software’ consists of Internet Explorer and Adobe Acrobat Reader. You can install these later if you need to do so.

- (b) The launch window (Figure 1.1) has buttons for installing Mathcad and ‘other software’. Click on the upper button to initiate the installation of Mathcad.
- (c) The InstallShield Wizard starts up, and the next several windows in the process have ‘InstallShield Wizard’ in the title bar. The following table indicates what action is needed within each of these windows.

Window	Response required
Welcome to the InstallShield Wizard for ...	Click on the Next> button.
License Agreement	Click on ‘I accept the terms in the license agreement’, and then click on Next> .
Customer Information	User Name: Type your own name. Organization: Type OU. Product Code: Enter your individual product code, as obtained from your OU StudentHome web page. (If possible, copy and paste the product code. Note however that this will have to be done in three parts, since there are three boxes to fill in, containing respectively 14, 4 and 4 digits or letters.) Finally, click on Next>.
Destination Folder	Click on Next> , unless you wish Mathcad to be installed in a location other than the default proposed.
Ready to Install the Program	Click on Install .

You are asked to read the license agreement carefully. However, this is hardly practical, given its length. The agreement has been accepted by the OU, and is not thought to contain any unreasonable conditions.

Installing Mathcad ...	The installation and configuration of the Mathcad software is now carried out. As the window says, this may take several minutes.
InstallShield Wizard Completed	The installation is now complete, but the licensing remains to be done. To acquire a license straight away, click on the box for ‘ R un the License Setup wizard to acquire/specify a license’. Otherwise, leave this box unticked. Either way, click on Finish to exit from the InstallShield Wizard.

If you do not acquire a license now, you will be asked to do so on launching Mathcad for the first time.

- (d) If you decided to run the License Setup Wizard straight away, then (after a pause) you will see the first of a sequence of windows titled ‘Mathcad License Setup’. Otherwise, you will see this window after clicking **Y**es in the dialogue box when first launching Mathcad, and should return to the following instructions at that point. The action needed within each window of the sequence is indicated in the table below.

Window	Response required
Welcome to Mathcad ...	The text in this window notes that you will need to be connected to the internet to complete all of the steps in this wizard. Click on Next> .
Setup Mechanism	The option for ‘Retrieve a license over the Internet (preferred)’ should already be checked. Ensure you have an internet connection and click on Next> .
Mathcad Product Code	The option for ‘Specify a product code below to obtain a license’ should be checked. Your product code (as entered previously during the installation process) should already be entered in the boxes below. Click on Next> .
Create Account	Click on the Create Account button. This takes you to an external web page, to create a PTC account.

The alternative described in this window of requesting a license file via email is not a valid option for OU students.

This web page appears in a window headed ‘Create a PTC.com account’.

You are required here to fill in the following **Personal Information**:

First Name, Last Name, Title, E-mail, Telephone, Company Name, Address, City, Postal Code (optional), Country, Preferred Language.

You need to give your correct email address, but may prefer to enter the following in lieu of further personal information.

First Name: MST121
 Last Name: Student
 Title: OU
 Telephone: 123456789
 Company Name: OU
 Address: OU
 City: Milton Keynes
 Country: United Kingdom

Click on 'No' beside 'I would like to receive e-mail from PTC' unless you wish to receive such email (there is no need to do so in connection with your OU studies).

Under **Security Information**, you are asked to specify a Password, with a Hint Question and Hint Answer.

Once all of this information has been provided, click on **Create Account>**. When a page stating that your account has been created is displayed, close your web browser and return to the License Setup Wizard, which should still be running.

Window	Response required
Create Account	With your PTC account now created, click on Next> .
Account Information	Enter your email address and the password you created on the PTC web page, then click on Finish> . The License Setup Wizard will contact the PTC server to download your license file.
You Are Done!	Click on OK to exit from the License Setup Wizard.

- (e) Now remove the CD for Mathcad from the DVD/CD drive and store it safely, away from heat, light and moisture.

Comment

- ◇ Your copy of Mathcad is supplied through a special arrangement between the Open University and the manufacturer of Mathcad. The terms of this agreement mean that the Open University, not the manufacturer, is responsible for supporting your use of Mathcad. For this reason, you should *not* contact the manufacturer of Mathcad directly (other than for the licensing process, as described above).

1.2 Installing the files and MST121 software

There is no need for internet access in this second stage of software installation.

This is the second (and final) stage of the software installation process. It is a good deal more straightforward and quicker than the first.

Activity 1.2 Installing the files and MST121 software

To install the files and MST121 software, you will need to have the MST121/MS221 Software CD.

The desktop should be visible on your computer screen. If there are any applications running, then close them now.

Depending on which version of *Windows* you have, at some stages of the installation process a window may appear asking whether you wish to allow the program to run. If this happens, then you should click to allow it to do so.

- (a) Put the MST121/MS221 Software CD into the DVD/CD drive. The installation process should start automatically, in which case the window shown in Figure 1.2 will appear.

If this does not happen, adopt one of the alternative approaches described in Activity 1.1(a), in order to obtain the window shown in Figure 1.2.

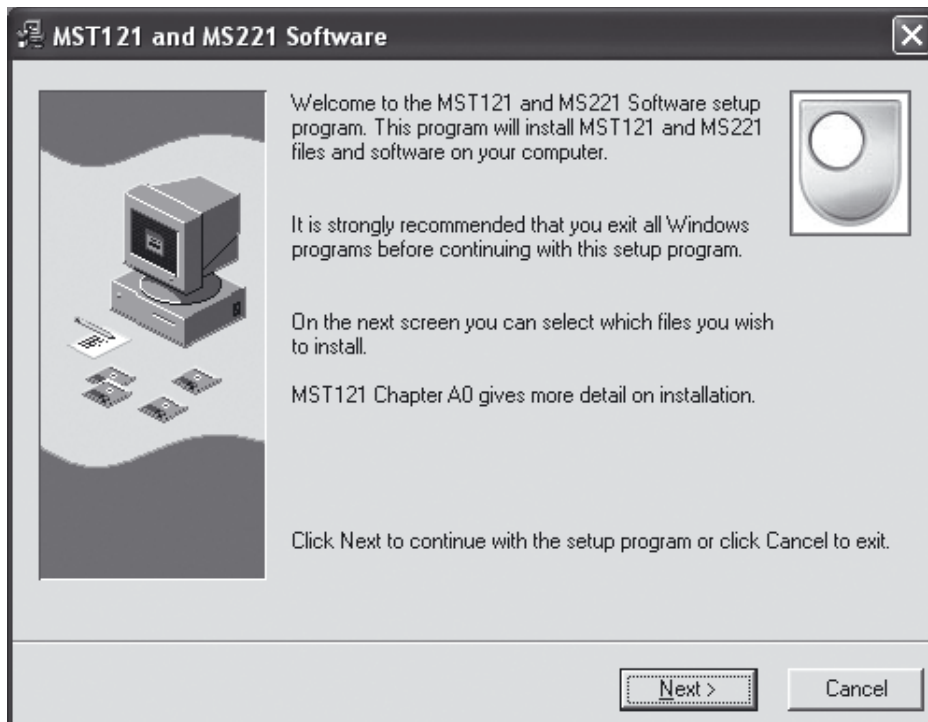


Figure 1.2 'Welcome' window

- (b) Click **N**ext.
- (c) Select the module(s) that you are studying currently by clicking on the appropriate check box(es). Then click **N**ext.
- (d) You will then be asked where on the computer you want to install the items for the module(s) you have selected. You should accept the default destination folders offered, by clicking on **N**ext, unless you deliberately want to arrange things differently.
- (e) A summary of what you have chosen to install is presented. Click **N**ext to begin the installation or **B**ack to make changes.
- (f) After clicking **N**ext, the items will be installed automatically. A dialogue box indicating a successful installation will appear when the installation is complete. Click on the **F**inish button to finish.
- (g) Now remove the MST121/MS221 Software CD from the DVD/CD drive and store it safely, away from heat, light and moisture.

If you click on the wrong box, then clicking again on the box will cancel the selection and empty the box.

Comment

- ◇ As a by-product of this installation process, you should find that an icon for Mathcad has appeared on your desktop. (Details of using this icon to run Mathcad will be given in Section 2.)

If the software for MST121 has been installed successfully, there will be two further new icons on your desktop, for the statistics programs **MST121 Simulations** and **MST121 OUStats**. Details of these programs and how to use them will be provided later, in the Computer Book for Block D.

- ◇ All of the MST121 files should have been placed in the folder **Documents\MST121** (unless you chose a destination folder other than the default for the files), and the folder structure on your hard disk will be similar to that shown in Figure 1.3. There are folders for most, but not all, of the MST121 chapters.

If you have installed the files for MS221, then there will be a similar folder structure based on **Documents\MS221**.

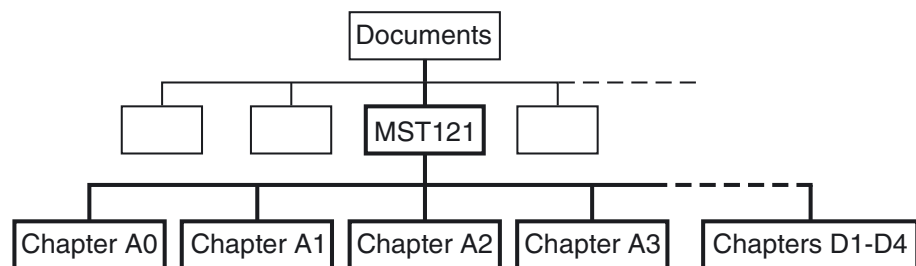


Figure 1.3 MST121 folder structure

If you wish, this structure can be viewed using ‘Computer’ or ‘My Computer’ (via **start** or an icon on the *Windows* desktop) or the *Windows* Explorer program.

If you look deeper, then you will find that the folder **Chapter A0** contains five files; these are the introductory files that you will be using shortly. The other folders contain further Mathcad files and data for the statistics programs for Block D of MST121. The folder structure is necessarily highly organised and hierarchical in nature. As long as you keep to the suggested structure, you will be able to store and find files as needed throughout MST121.

Everything should now be ready for you to start the introductory Mathcad sessions in Section 2. From now on you will use the copies of Mathcad and the files that are stored on your computer’s hard disk. If anything goes wrong with these files during the activities, or during your own independent experimentation (which is encouraged), then you can re-install them from the MST121/MS221 Software CD as described above.

Summary of Section 1

In this section, you should have:

- ◇ installed Mathcad;
- ◇ installed the MST121 software and files for the module(s) you are studying.

2 Introduction to Mathcad

2.1 First steps

Mathcad is a powerful and expressive mathematical tool, and you will use it frequently throughout MST121. It has many features, some of them quite technical, which will be introduced gradually in the context of the teaching material. The rest of this chapter introduces some of the basic Mathcad facilities. You will also learn the conventions and working practice that MST121 adopts when using Mathcad.

If you have not continued straight on from Section 1, then switch on your computer.



Activity 2.1 Your first calculation

- Run Mathcad by double-clicking on the shortcut icon labelled **Mathcad 14** that should have appeared on your desktop after installation from the second CD.
(Alternatively, open the **start** menu; then click on **All Programs**, **Mathcad**, **Mathcad 14**, and finally select the **Mathcad 14** item.)
- When Mathcad first opens, you will see a window entitled 'Mathcad Resources: Welcome ...'. Close this down by clicking on the 'Close' button at the top right of the window. (This window refers to sources of information that you may wish to access later, but there is no need to do so for the moment. Your introduction to Mathcad for MST121 and/or MS221 is contained in this chapter.)

The main Mathcad window should now be visible, as shown in Figure 2.1.

It is assumed that you have by now carried out both of the software installation processes described in Section 1, one for each of the CDs provided.

Alternatively, select its **File** menu and **C**lose.

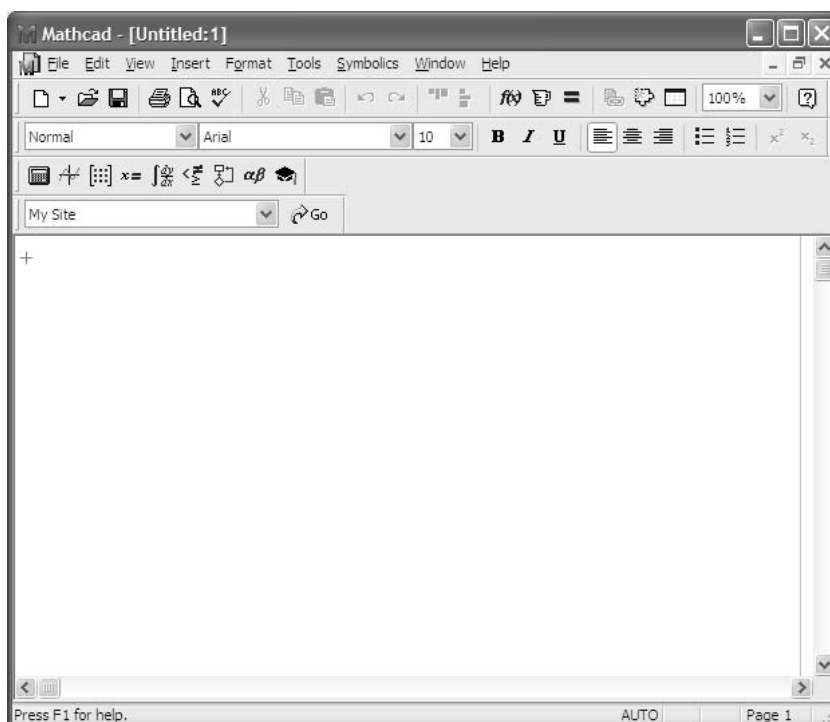
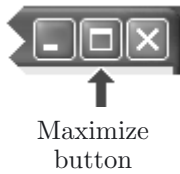


Figure 2.1 The opening Mathcad window



- (c) If the Mathcad window does not fill your screen, you can use the ‘Maximize’ button to expand it. (This is the middle one of the three buttons in the top right-hand corner of the window. Alternatively, press the [Alt] key and the spacebar together, then choose **Maximize** from the menu that appears.)

Mathcad automatically starts with a new, blank worksheet.

Depending on the size of your screen, you may see a solid vertical line indicating the right-hand margin of the worksheet page.

- (d) You will see a small red cross cursor in the top left-hand corner of the worksheet. Move the mouse arrow to somewhere in the middle of the worksheet page, and click the mouse button to move the red cross cursor to that point.
- (e) Now you can type in your first calculation. You need to follow the typing instructions exactly. In particular, you *must not use the spacebar* to obtain extra spaces in an expression. These are not needed, and using the spacebar may have unwanted side-effects.

Carefully type the following.

131+285=

On the screen you will see

131 + 285 = 416

- (f) The blue editing lines are positioned around the number 285, to allow you to extend the calculation (for example, by adding another number). To finish the calculation, remove the blue editing lines by moving the mouse arrow to a fresh point in the worksheet and clicking the mouse button. You can also finish the calculation by pressing the [Enter] key (which may alternatively be labelled [Return] or [↵]).

Comment

- ◇ The underlined letters shown in the menu headings of Figure 2.1 may not initially be visible to you. However, they should become visible if you press the [Alt] key (and disappear if you press [Alt] again).

They provide an alternative to use of the mouse when accessing menus. For example, [Alt]f (that is, holding down the [Alt] key and typing f) will open the **F**ile menu, [Alt]e will open the **E**dit menu, and so on. Once a menu is open, a similar approach with [Alt] and an underlined letter can be used to select items from the menu, and wherever underlines appear, this alternative can be applied. To facilitate this, the text includes all such underlines where appropriate, but these may be ignored if you prefer to click with a mouse.

- ◇ These underlines can be made to appear at all times, if so desired, as follows. In recent versions of *Windows* open the **s**tart menu, then click on **C**ontrol **P**anel. Click on **E**ase of **A**ccess then **M**ake the **k**eyboard **e**asier to use. Check the box for ‘Underline keyboard shortcuts and access keys’, and click **S**ave or **O**K. In earlier versions of *Windows* open the **s**tart menu; then click on **C**ontrol **P**anel and **D**isplay. In the ‘Display Properties’ dialogue box, choose the ‘Appearance’ tab and click on the **E**ffects... button. Then uncheck the box for ‘**H**ide underlined letters for keyboard navigation until I press the Alt key’. Following this, click on **O**K twice (to close two successive boxes), then close the ‘Control Panel’ window.

This is a general feature of the *Windows* environment, rather than being specific to Mathcad.

There is no need to keep this worksheet containing a single calculation, so the next step is to dispose of it.

Activity 2.2 Closing a worksheet

- Click on **File** in the menu bar, and select **C**lose from the menu. This will bring up a 'Save changes' dialogue box.
- Click on the **N**o button to dispose of the worksheet. The main body of the Mathcad window will turn grey, since now there is no worksheet open.

The next task is to create a new worksheet.

Activity 2.3 Creating a new worksheet

- Click on the **F**ile menu, and select **N**ew...
(Alternatively, type [Ctrl]n, that is, hold down the control key [Ctrl] and then press n, or click on the **N**ew button, which is just below the menu bar, at the left-hand end of the standard toolbar. Both of these alternative methods create a new (Normal) worksheet immediately, and do not require step (b) below.)
- This will bring up the 'New' dialogue box, containing a list of worksheet templates. In this list, **Normal** should be selected by default; if not, click on it. Then click on the **OK** button to create a new (Normal) worksheet. The drop-down menu, obtained by clicking on the triangle to the right of the 'New' button, gives a similar choice.
- Now try some calculations of your own!

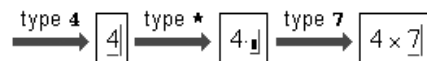
Use +, -, the asterisk * for multiplication, and the forward slash / for division. You can use the keys either on the main keyboard, where * is given by [Shift]8, or on the numeric keypad. (Note that [Num Lock] must be *on* to use the numbers and operators on the numeric keypad.)

Although you type * for multiplication in Mathcad, this appears on the screen initially as a small raised dot, changing to a multiplication sign × as you continue typing.

For example, if you type

4*7

then on the screen you will see



Remember to type an equals sign = at the end of each calculation, to see the answer.



New button

When [Num Lock] is on, the word 'NUM' will appear near the right-hand end of the status bar, at the bottom of the Mathcad window.

To save the worksheet created in Activity 2.3 as a file, proceed as follows.

Activity 2.4 Saving your own worksheet

If you installed only the MS221 files, then locate the **MS221** folder.

If in Activity 1.2(c) you installed the files in a non-default destination, then you will need to browse to locate the appropriate folder.

- (a) Click on the **F**ile menu, and select **S**ave **A**s... .
This will bring up the standard *Windows* ‘Save As’ dialogue box.
- (b) Locate the **MST121** folder (using the scroll bar if necessary), and double-click on it.
The **MST121** folder contains a folder for each chapter. Double-click on the folder for **Chapter A0**, and the contents of the ‘Save As’ dialogue box will change once again, to show icons for the five files for Chapter A0.
- (c) Now save your worksheet in the **Chapter A0** folder as follows. Type **my first calculations** (or any other acceptable name, using letters, numbers and spaces) in the **F**ile **n**ame box, towards the bottom of the dialogue box. The entry in the **S**ave **a**s **t**ype box below should already say ‘Mathcad XML Document (*.xmcd)’, and there is no need to alter this. Finally, click on the **S**ave button.

Now that you have saved your worksheet, you can go on to print it. Note that it is always good practice to save a worksheet *before* printing it, in case you have a problem with printing.

Activity 2.5 Printing a worksheet

Make sure that your printer is switched on, and has some paper loaded!

- (a) Click on the **F**ile menu, and select **P**rint... .
This will bring up the ‘Print’ dialogue box.
- (b) Choose whether to print all of your worksheet or only particular pages.
- (c) Then click on the **OK** button to print.

Comment

- ◇ You will need to supply Mathcad printouts for most tutor-marked assignments, and these might include suitably amended pages or excerpts from the files supplied. You might also choose to print out pages which you find difficult to read on the screen – for example, those with a large amount of text. It is important therefore to have a working printer set up for your computer.

Should you experience problems with printing from Mathcad, consult the MST121 website where up-to-date advice is given.

The final topic of this subsection is how to terminate a Mathcad session. If you have altered a worksheet or created a new one, then you should save it first, as described above.

Activity 2.6 Exiting from Mathcad

- (a) Click on the **F**ile menu, and select **E**xit.
- (b) If you followed the earlier advice on saving changes to worksheets, then Mathcad will disappear and you will return to the desktop.
If, however, you have made unsaved changes, then you will see the same ‘Save changes’ dialogue box that was referred to in Activity 2.2.
If you want to save the changes, click on the **Y**es button and follow the instructions in Activity 2.4.
Otherwise, click on the **N**o button.

Even if you wish to move straight on to Subsection 2.2, it is a good idea to learn how to exit from Mathcad at this point.

You have now finished your first Mathcad session, and can close down the computer in the usual way, if you wish.

2.2 Worksheets, expressions and text

Now it is time for you to start working through the introductory files. These contain detailed Mathcad worksheets with full comments. You should work through them carefully, as they are the basis for later work.

Begin the session by running Mathcad as described in Activity 2.1(a).



When opening Mathcad for the second time, you will see the ‘Mathcad Tips’ window. This contains a tip about Mathcad use. You can see more tips by clicking on the **N**ext Tip button. When you have seen enough, click on the **C**lose button.

Without further action, the ‘Mathcad Tips’ window will appear each time that you open Mathcad. If you wish to prevent this then (before closing the window) click to uncheck the box for ‘**S**how tips on startup’.

If you subsequently wish to reinstate the ‘Mathcad Tips’ at startup, go to the **T**ools menu and click on **P**references.... In the ‘Preferences’ dialogue box, on the ‘General’ tab, check the box for ‘**S**how Mathcad tips at startup’.

Activity 2.7 Opening the first Mathcad file, 121A0-01

- (a) Click on the **F**ile menu and select **O**pen.... You will see a standard Windows ‘Open’ dialogue box.
- (b) Locate the **MST121** folder, and double-click on it. The **MST121** folder contains a closed folder for each chapter. Double-click on the folder for **Chapter A0** (which contains the files for this chapter), and the contents of the ‘Open’ dialogue box will change once again, to show icons for the five files for Chapter A0.
- (c) Now double-click on the name **121A0-01 Worksheets expressions and text** to open it. You will then see the first introductory worksheet.

If you installed only the MS221 files, then you will not be able to locate the files for MST121 Chapter A0.

We refer to what you see on the screen as a *worksheet*, to distinguish it from the *file*, which is what the computer stores on disk.

Once the file is open, it is good practice to make your own working copy of it, leaving the original copy on your hard disk unchanged.

Activity 2.8 Making your own working copy of a file

- (a) Click on the **File** menu, and select **Save As...**.
You will then see the standard *Windows* 'Save As' dialogue box, as in Activity 2.4(b).
- (b) Now edit the **File name** box to contain my **A0-01 Worksheets expressions and text** (or any acceptable new name that you prefer).
- (c) Click on the **Save** button.

Your copy of the first worksheet is now on the screen, and you can work through it. First, you will learn how to navigate worksheets. Then you will discover how to enter and edit mathematical expressions and text in Mathcad.

Activity 2.9 Worksheets, expressions and text

Work through the worksheet, bearing in mind the following general points.

1. There is quite a large amount of text in this and the other introductory Mathcad worksheets. If you find it difficult to read so much text on the screen, then you can print out some of the pages of a worksheet (as described in Activity 2.5) and read through the printed version, returning to the screen to perform the various tasks in the worksheet. Also, you may wish to print out some of the key pages giving Mathcad techniques, for later reference.
2. You can finish a computer session:
 - ◇ when you have worked through a whole worksheet;
 - ◇ when you want a rest (at any time);
 - ◇ when you want to work on something else.

The steps for closing a worksheet and exiting from Mathcad were given in Subsection 2.1.

3. When finishing, you are advised to save your *current* version of the Mathcad worksheet, which will then incorporate any alterations you have made to it during the session. If you wish to return to this worksheet, remember to open your copy of it, for example, my **A0-01 Worksheets expressions and text**, unless you want to see the original version, **121A0-01 Worksheets expressions and text**.

The next introductory file shows you, in more detail, how both mathematical expressions and text can be manipulated on the screen.

Activity 2.10 Rearranging worksheets

Locate in the **Chapter A0** folder the file **121A0-02 Rearranging worksheets**. Open it, save your own copy, and work through the worksheet.

Summary of Section 2

In this section, you should have obtained a preliminary idea of Mathcad's features, which will be developed further as MST121 progresses.

In particular, you have seen how to:

- ◇ run Mathcad and exit from Mathcad;
- ◇ create a new Mathcad worksheet;
- ◇ save, print and close a Mathcad worksheet;
- ◇ open a file and make your own working copy of it;
- ◇ navigate a Mathcad worksheet using the vertical scroll bar, [Shift] [Page Up], [Shift] [Page Down] and **Go to Page...**;
- ◇ enter and edit simple mathematical expressions and text;
- ◇ select, re-position and copy regions containing mathematical expressions or text, and tidy up the Mathcad screen.

3 Calculation

More detail on these topics can be found in the *Revision Pack*.

In this section, the main features of ‘numbers’ and ‘calculation’ are reviewed briefly, and you are shown how to calculate with Mathcad.

3.1 Arithmetic

The decimal system

The decimal system uses the **digits** 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 to represent numbers in terms of powers of 10. For example,

$$12.345 = 1 \times 10 + 2 \times 1 + 3 \times \frac{1}{10} + 4 \times \frac{1}{100} + 5 \times \frac{1}{1000};$$

that is,

$$12.345 = 1 \times 10^1 + 2 \times 10^0 + 3 \times 10^{-1} + 4 \times 10^{-2} + 5 \times 10^{-3}.$$

The position of a given digit in the number relative to the decimal point indicates the power of 10 which that digit multiplies. The positions to the right of the decimal point are referred to as ‘decimal places’; for example, in 12.345, the digit 5 is in the third decimal place.

The above number is a **finite decimal**; that is, it has only finitely many decimal places. Most calculations are carried out with finite decimals, but there are many quantities whose size cannot be described *exactly* with finite decimals. For example, the decimal representations of $\frac{1}{3}$ and $\sqrt{2}$, namely $\frac{1}{3} = 0.333\dots$ and $\sqrt{2} = 1.414\dots$, both ‘go on forever’. Therefore we also need types of numbers whose decimal representations are **infinite decimals**; such decimals have infinitely many decimal places.

The table below classifies various types of numbers, with examples of each. In the column on the left are the names of these types of numbers (with various alternative names in brackets), and the mathematical symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ used to denote the corresponding sets of numbers. For example, $\mathbb{N}$ denotes the set of all natural numbers: 1, 2, 3, $\dots$. (When writing these symbols, just do the best you can!)

Table 3.1 Types of numbers

Number type	Examples
natural numbers, $\mathbb{N}$ (counting numbers, positive integers)	1, 2, 3
integers, $\mathbb{Z}$ (whole numbers)	$-3, -2, -1, 0, 1, 2, 3$
rational numbers, $\mathbb{Q}$ (fractions)	$\frac{1}{2} = 0.5, \frac{1}{3} = 0.333\dots, \frac{3}{1} = 3, -\frac{9}{11} = -0.8181\dots$
real numbers, $\mathbb{R}$ (decimals – finite and infinite)	$\frac{1}{2} = 0.5, \frac{1}{3} = 0.333\dots, \sqrt{2} = 1.4142\dots, \pi = 3.1415\dots$

Remember that if a is non-zero, then

$$\begin{aligned} a^0 &= 1, \\ a^{-1} &= 1/a, \\ a^{-2} &= 1/a^2, \end{aligned}$$

and so on.

The symbol ‘ $\dots$ ’ means ‘and so on’. It indicates that something has been omitted whose nature can be deduced from the context.

There follow some observations about this table.

- ◇ The four types of number in the table are increasingly general: every natural number is an integer, every integer is a rational number, and every rational number is a real number.
- ◇ The symbol $\mathbb{Z}$ derives from the German word *Zahl*, meaning ‘number’, $\mathbb{Q}$ derives from the word ‘quotient’, and $\mathbb{R}$ derives from the word ‘real’.
- ◇ A **recurring decimal** is one for which a particular digit or block of digits is repeated endlessly to the right. For example: $\frac{1}{3} = 0.333\dots$ and $\frac{29}{54} = 0.5370370\dots$. Two notations in common use for recurring decimals are indicated below:

$$\frac{1}{3} = 0.\dot{3} = 0.\overline{3} \quad \text{and} \quad \frac{29}{54} = 0.5\dot{3}7\dot{0} = 0.5\overline{370}.$$

These recurring decimal representations can be found by long division or with a calculator (up to the accuracy of the calculator).

- ◇ In fact, all rational numbers have finite or recurring decimal representations, and all numbers with finite or recurring decimal representations are rational. Also, finite decimals may be considered as recurring, since we can include the so-called ‘trailing zeros’ which are usually omitted. For example: $0.5 = 0.5000\dots$
- ◇ Real numbers that are not rational are called **irrational numbers**; these are the ones with non-recurring decimal representations. For example, it can be shown that $\sqrt{2}$ and π are both irrational.

You can see from these examples that numbers can be represented in more than one way. For example, $\frac{1}{2} = 0.5$. Therefore, when we perform calculations, we may have to decide which of these representations to use. For example, faced with $\frac{1}{3} + \frac{1}{6}$, it is more straightforward to use the rules for fractions,

$$\frac{1}{3} + \frac{1}{6} = \frac{2}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2},$$

rather than adding the infinite decimals,

$$\frac{1}{3} + \frac{1}{6} = 0.333\dots + 0.1666\dots$$

The word ‘real’ distinguishes these numbers from other numbers – for example, complex numbers, studied in MS221 – which are sometimes considered to be ‘unreal’.

MST121 uses the dot notation for recurring decimals.

$\frac{1}{3}$ and $\frac{2}{6}$ are equivalent fractions. Fractions with the same denominator, like $\frac{2}{6}$ and $\frac{1}{6}$, are added by adding the numerators.

The real line

The various representations of numbers each have their own advantages and disadvantages. For example, the decimal representation is convenient when we wish to locate real numbers on a number line (such as the x -axis of a graph). On such a line, the integers are represented by equally spaced points, appropriately labelled and increasing from left to right, as in Figure 3.1.

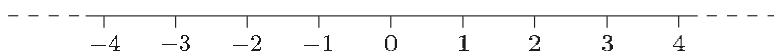


Figure 3.1 A number line

The segment of the number line between each adjacent pair of integers is divided into 10 equal segments, the division points being labelled with corresponding tenths. Each of these 10 segments is then itself divided into 10 equal segments, the division points being labelled with corresponding hundredths. This is illustrated in Figure 3.2.

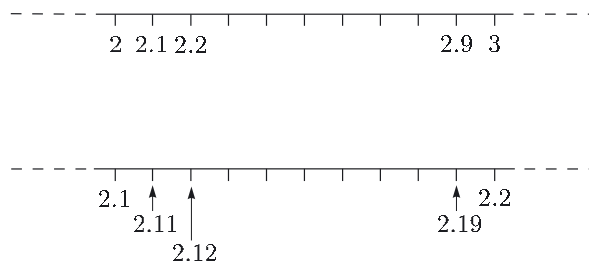


Figure 3.2 Dividing into ten segments

We imagine this process being repeated indefinitely. Then any finite decimal can be located, at least in principle, on the number line. For example, 1.414 lies between 1.41 and 1.42, four tenths of the way along.

Even *infinite* decimals can be located in this way. For example, the unique point corresponding to $\frac{1}{3} = 0.333\dots$ lies to the right of each of the finite decimals

$$0, 0.3, 0.33, 0.333, 0.3333, \dots,$$

and to the left of each of the finite decimals

$$1, 0.4, 0.34, 0.334, 0.3334, \dots$$

In a similar way, you can locate $\sqrt{2} = 1.414\dots$ on the number line.

We refer to a line which represents all possible real numbers in this way as the **real line**.

Decimal representation facilitates the comparison of real numbers. For example, the numbers $\sqrt{2}$ and $\frac{10}{7}$ have decimal representations $\sqrt{2} = 1.414\dots$ and $\frac{10}{7} = 1.428\dots$. These differ first in the second decimal place. This indicates that $\sqrt{2}$ is less than $\frac{10}{7}$, and so lies to the left of $\frac{10}{7}$ on the real line. We write ' $\sqrt{2}$ is less than $\frac{10}{7}$ ' in symbols as

$$\sqrt{2} < \frac{10}{7} \quad \text{or, equivalently,} \quad \frac{10}{7} > \sqrt{2}.$$

The latter form is read as ' $\frac{10}{7}$ is greater than $\sqrt{2}$ '.

Arithmetic and rounding

The four basic arithmetic operations,

$$+ \text{ (addition), } - \text{ (subtraction), } \times \text{ (multiplication), } \div \text{ (division),}$$

can be applied to pairs of real numbers to yield a real number as the answer – the only exception to this is 'division by 0', which is not defined. For example,

$$1 + 2 = 3, \quad 3 - 4 = -1, \quad 5 \times (-6) = -30, \quad 7 \div 8 = 0.875.$$

Calculations involving more than one of the basic operations often require brackets to avoid ambiguity. For example,

$$1 - (2 + 3) = -4, \quad \text{whereas} \quad (1 - 2) + 3 = 2.$$

The above calculations can all be performed by hand, without help from a calculator or computer, and the results are exact. But many calculations in which we want the answer as a decimal are most conveniently done with a calculator or computer, and in such cases the answer presented is often not exact. For example, using a calculator, we obtain

$$\frac{98}{78} = 1.256\,410\,256\,4 \quad (\text{to 10 d.p.}).$$

This answer is not exact, but is rounded to ten *decimal places* (d.p.); that is, the answer presented is the number with ten decimal places

Similarly, the symbols $\leq$ and $\geq$ are read as 'less than or equal to' and 'greater than or equal to', respectively. Thus $6 \leq 6$ and $6 \leq 7$ are both true statements.

Alternative notations are

$$5 \times (-6) = 5 \cdot (-6), \\ 7 \div 8 = 7/8 = \frac{7}{8}.$$

The fraction $\frac{98}{78}$ has a recurring decimal representation.

which is nearest to the exact answer. Recall that we round to a given number of decimal places by keeping the digits up to that decimal place, and if the first discarded digit is 5 or more, increasing the last digit kept by 1. For example,

$$1.364 = 1.36 \text{ (to 2 d.p.)} \quad \text{and} \quad 1.367 = 1.37 \text{ (to 2 d.p.)}.$$

Different calculators may round to different numbers of decimal places. For example, on a calculator which rounds to eight decimal places, we would obtain

$$\frac{98}{78} = 1.25641026 \text{ (to 8 d.p.)},$$

since the digit in the 9th decimal place of 1.2564102564 is 6. You will see shortly that Mathcad allows you to *choose* the number of decimal places to which displayed answers are rounded.

We often choose to round answers correct to a particular number of *significant figures* (s.f.), rather than to a particular number of decimal places. This means that the answer presented is the number with the given number of digits (counting from the first non-zero digit) which is nearest to the exact answer. (The rule for the last digit is the same as when rounding to a number of decimal places.) For example,

$$\frac{98}{78} = 1.2564 \text{ (to 5 s.f.)} \quad \text{and} \quad \frac{82}{7} = 11.714 \text{ (to 5 s.f.)}.$$

The Appendix contains guidelines on how to present numerical answers to questions.

Powers

Another situation in which we often need help from a calculator or computer is in calculating powers.

Other words for ‘power’ are ‘exponent’ and ‘index’.

First, here is a reminder of some of the rules for handling *integer* powers. If a is a non-zero real number, and p and q are integers, then

$$a^p \times a^q = a^{p+q} \quad \text{and} \quad (a^p)^q = a^{pq}. \quad (3.1)$$

For example, with $a = 2$, $p = 3$ and $q = -4$,

$$2^3 \times 2^{-4} = 2^{3-4} = 2^{-1} = \frac{1}{2} \quad \text{and} \quad (2^3)^{-4} = 2^{3 \times (-4)} = 2^{-12} = \frac{1}{4096}.$$

Next, more general powers are discussed. Suppose that a is a *positive* number. If n is a positive integer, then $\sqrt[n]{a}$ denotes that *positive* number which, when raised to the power n , gives the answer a ; it is called the ***n*th root** of a . For example, $\sqrt[3]{8} = 2$ because $2^3 = 8$. Also recall that, for any positive integer n , $\sqrt[n]{0} = 0$. Most roots cannot be found easily by hand calculation, however.

For $n = 2$, this is the **square root** of a , written $\sqrt{a}$.

A more convenient notation for the n th root is $a^{1/n}$. This notation has two important advantages.

- ◇ It is consistent with the rules for integer powers stated in equation (3.1). For example, on using the second of these rules, with $a > 0$, $p = 1/n$ and $q = n$, we obtain

$$(a^{1/n})^n = a^{(1/n) \times n} = a^1 = a,$$

as expected.

- ◇ It can be extended to allow any power of a *positive* number a . First, if m and n are integers, with n positive, then

$$a^{m/n} = (\sqrt[n]{a})^m \quad \text{or} \quad a^{m/n} = \sqrt[n]{a^m}.$$

For example,

$$8^{4/3} = (\sqrt[3]{8})^4 = 2^4 = 16 \quad \text{or} \quad 8^{4/3} = \sqrt[3]{8^4} = \sqrt[3]{4096} = 16.$$

These two definitions give the same value for $a^{m/n}$.

This extension is covered in modules on real analysis.

This defines a^x when x is a rational number m/n . The extension to values of x which are arbitrary real numbers (such as $2^{\sqrt{2}}$, which corresponds to $a = 2$ and $x = \sqrt{2}$) can be carried out by using approximations to x by rational numbers.

With these extensions, the rules for powers in equation (3.1) continue to hold, as do rules such as $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

Scientific notation

Before you try some calculations involving powers, it is important to recall a variation of decimal notation which is used for dealing with numbers that are either very large or very small. This is **scientific notation** – also known as **exponential notation** and **standard form**. Scientific notation takes the following form:

(a number between 1 and 10, not including 10) $\times$ (a power of 10).

For example,

$$253 = 2.53 \times 10^2, \quad 0.253 = 2.53 \times 10^{-1}.$$

The appropriate power of 10 is found by counting how many positions the decimal point has to be moved to obtain the required ‘number between 1 and 10’. The sign of that power is positive if the decimal point is moved to the left, and negative otherwise. Different calculators apply different thresholds at which they change from displaying an answer in ordinary decimal notation to scientific notation. Mathcad allows you to *choose* this threshold.

Calculation with and without Mathcad

In the next two activities, you are invited to do some calculations by hand or with your calculator, and then with Mathcad.

Activity 3.1 Arithmetic and rounding

Use hand calculation or your calculator, as appropriate, to check that the following results are correct.

- (a) $625^{3/4} = 125$
- (b) $\frac{78}{98} = 0.795\,918$ (to 6 d.p.)
- (c) $2^{4/3} = 2.520$ (to 3 d.p.)
- (d) $2^{100} = 1.27 \times 10^{30}$ (to 3 s.f.)

Solutions are given on page 41.

In rounded numbers, trailing zeros are sometimes omitted. So part (c) could be written as

$$2^{4/3} = 2.52 \text{ (to 3 d.p.)}.$$

The next activity leads you through the process of entering arithmetic expressions in Mathcad and evaluating them. You will also see how to choose the number of decimal places that Mathcad uses to display numerical answers, and the threshold for its use of scientific notation.



Switch on your computer and run Mathcad.

Activity 3.2 Calculating with Mathcad

Locate in the **Chapter A0** folder the file **121A0-03 Calculating with Mathcad**. Open it, save your own copy, and work through the worksheet.

Comment

This activity should convince you that, for a single calculation, it is usually easier to use hand calculation or your calculator. Mathcad will be more effective when you are faced with many calculations of a similar type, and will also enable you to retain a record of your calculations.

3.2 Variables

A powerful feature of mathematics is its use of symbols (usually letters or combinations of letters) to represent numbers. We use symbols to represent numbers that may take various values; that is, they may vary. Such a symbol is called a **variable**.

Variables are often related to other variables. For example, the formula for the circumference of a circle of a given radius is

$$C = 2\pi r.$$

Here r is a variable that denotes the radius of the circle, and C is a variable that denotes the circumference of the circle. The variables r and C can take only positive real values. The number $\pi = 3.141\,592\,6\dots$ does not vary, so it is not a variable. Given any positive radius r , the formula enables you to calculate the corresponding circumference C . Since r can take *any* positive value, it is called an **independent variable** in this formula. On the other hand, C is a **dependent variable**, since its value depends on that of r . Similarly, in the formula for the area of a triangle

$$A = \frac{1}{2}bh,$$

the independent variables b and h represent the lengths of the base and height of the triangle, and the dependent variable A represents the triangle's area.

The next two activities ask you to evaluate several formulas for particular values of the independent variables, and show you how to work with variables in Mathcad.

In the discussion of powers, the variables a , p , q , m , n and x were used.

In Mathcad and on your calculator, the number π is a built-in constant.

Activity 3.3 Using variables

Use your calculator to find the values of the dependent variables from each of the following formulas. Give your answers correct to three significant figures.

- (a) The circumference C of a circle of radius r :

$$C = 2\pi r, \quad \text{for } r = 2 \text{ and for } r = 5.$$

- (b) The area A of a circle of radius r :

$$A = \pi r^2, \quad \text{for } r = 1.5 \text{ and for } r = 3.7.$$

- (c) The volume V of a cylinder of height h whose ends are circles of radius r :

$$V = \pi r^2 h, \quad \text{for } r = 1.5, \ h = 8.$$

- (d) The hypotenuse c of a right-angled triangle with sides a, b, c :

$$c = \sqrt{a^2 + b^2}, \quad \text{for } a = 1, \ b = 2 \text{ and for } a = 3, \ b = 4.$$

Solutions are given on page 41.



Switch on your computer and run Mathcad (if necessary).

Activity 3.4 Using variables in Mathcad

Locate in the **Chapter A0** folder the file **121A0-04 Using variables**. Open it, save your own copy, and work through the worksheet.

Comment

If a variable is redefined within a worksheet, Mathcad (by default) places a green squiggle beneath the name of the variable that has been redefined. This feature may be turned off (or reinstated) with the key combination **[Ctrl] [Shift] r**.

This activity should convince you that if you need to calculate the value of a dependent variable for *many* values of the independent variables, then using Mathcad is very efficient.

The earlier Mathcad worksheets in this chapter used the $=$ sign to evaluate arithmetic expressions numerically. This worksheet introduced the symbol $:=$ which is used in Mathcad to define variables and assign values to them. For example, the Mathcad expression $a := 3$ assigns the value 3 to the variable a . When *writing* mathematics, this appears as a sentence such as ‘Now define $a = 3$.’

Summary of Section 3

This section recalled basic properties of numbers and arithmetic, introduced some new notation for several types of numbers, and showed you how arithmetic is done with Mathcad. In particular, you met:

- ◇ the set of *natural* numbers, $\mathbb{N}$: $1, 2, 3, \dots$;
- ◇ the set of *integers*, $\mathbb{Z}$: $\dots, -3, -2, -1, 0, 1, 2, 3, \dots$;
- ◇ the set of *rational* numbers, $\mathbb{Q}$: all numbers of the form m/n , where m, n are integers and $n \neq 0$;
- ◇ the set of *real* numbers, $\mathbb{R}$: all finite or infinite decimals, which together comprise the *real line*;
- ◇ formulas giving the value of a dependent variable in terms of independent variables;

and you saw:

- ◇ that rational numbers are equivalent to *finite* or *recurring* decimals;
- ◇ rules which apply to powers and the meaning of rational powers;
- ◇ rounding numbers and scientific notation;
- ◇ how to enter and evaluate arithmetic expressions in Mathcad.

Exercises for Section 3

Exercise 3.1

- (a) Use hand calculation or your calculator, as appropriate, to evaluate the following expressions, giving your answers to three significant figures.

(i) $49^{3/2}$ (ii) $50^{3/2}$ (iii) $\frac{26}{65}$ (iv) $\frac{27}{65}$ (v) 3^{-10}

- (b) Use Mathcad to confirm your answers to part (a).

To use Mathcad, you will need to create a new (Normal) worksheet.

Exercise 3.2

- (a) Use your calculator to find the values of the dependent variable from each of the following formulas, giving your answers to three significant figures.

- (i) The volume V of a sphere of radius r :

$$V = \frac{4}{3}\pi r^3, \quad \text{for } r = 2.5 \text{ and for } r = 4.7.$$

- (ii) The semi-perimeter s and area A of a triangle with sides a , b and c :

$$s = \frac{1}{2}(a + b + c) \quad \text{and} \quad A = \sqrt{s(s-a)(s-b)(s-c)},$$

for $a = 2$, $b = 3$, $c = 4$.

- (b) Use Mathcad to confirm your answers to part (a)(i).

4 Algebra

The word ‘algebra’ comes from the Arabic *al-jabr*, meaning ‘the reunion of broken parts’. Its usage stems from its inclusion in the title of a book by Mohammed bin Mûsâ al-Khowârizmî, written in Baghdad circa 825 AD.

More detail on these topics can be found in the *Revision Pack*.

When variables are used in problems, we obtain expressions and equations which involve these variables, linked together by precisely the same operations that are used with numbers; see, for example, the formulas in Activity 3.3. Such expressions and equations can often be rearranged in useful ways to give equivalent expressions and equations. This process of rearrangement is called **algebra**.

Algebra has been crucial to the development of mathematics throughout most of its history. It permits mathematical statements to be expressed in concise and general ways. What is more, one algebraic statement may lead on to another, which is equivalent to the first, but provides a different emphasis and hence permits a new understanding of the underlying situation.

In this section, the basics of algebra are reviewed, and you are shown how to do many basic algebraic operations with Mathcad.

4.1 Algebraic expressions

An **algebraic expression** is a single number, a single variable or a collection of numbers and variables which are connected together by the basic arithmetic operations $+$, $-$, $\times$, $\div$, and also by powers. For example, all of the following are (algebraic) expressions:

$$2ax - 3by + z, \quad 5x^2 - 3x - 2, \quad \frac{4}{3(x+2)} + \frac{2}{x+3}, \quad \sqrt{x^2 + y^2}.$$

In such expressions, we usually omit the multiplication sign $\times$ between two expressions unless these are both numbers, or unless we wish to emphasise that multiplication is being used. Also, as with numbers, the division sign $\div$ is usually replaced by $/$ or by a horizontal bar, and brackets are used to indicate parts of expressions which are to be evaluated together.

An expression often involves simpler expressions, known informally as **terms** of the original expression. For example, in the expression

$$2ax - 3by + z,$$

the expressions $2ax$, $-3by$ and z are terms.

An expression in which a number of terms are added together is called a **sum**. For example,

$$5x^2 - 3x - 2$$

is a sum with three terms, namely $5x^2$, $-3x$ and -2 . The term -2 is a constant term.

An expression in which a number of terms are multiplied together is called a **product**, and each of the terms is a **factor** of the product. If a particular term of a product is of interest, then the remainder of the product is called the **coefficient** of that term. For example, $2a$ is the coefficient of x in the product $2ax$.

An algebraic expression in which one expression, the **numerator**, is divided by another expression, the **denominator**, is called a **fraction**, or **quotient**. For example, the fraction

$$\frac{4}{3(x+2)}$$

has numerator 4 and denominator $3(x+2)$.

Equivalent algebraic expressions

Two algebraic expressions are **equivalent** if they take the same value for all possible values of the variables; this equivalence is indicated using an = sign. There are several basic ways of rearranging an algebraic expression to obtain equivalent expressions. Some of these methods aim to simplify the expression, whereas others aim to put the expression into a form which makes it more convenient for a particular technique to be applied.

Collecting like terms A basic way to simplify a sum (or a product) is to collect together terms of the same type. For example, the sum

$$x^2 + 2x + 1 + x^2 - 2x + 1$$

has terms in x^2 and x , and constant terms. Collecting together like terms, we obtain

$$\begin{aligned} x^2 + 2x + 1 + x^2 - 2x + 1 &= x^2 + x^2 + 2x - 2x + 1 + 1 \\ &= 2x^2 + 2. \end{aligned}$$

Here the terms $2x$ and $-2x$ cancel out.

Sums like these, which involve powers of a variable, are usually arranged with the powers in increasing or decreasing order. For example, we would normally rearrange $2x + x^2 - 3$ as $x^2 + 2x - 3$ or as $-3 + 2x + x^2$. There is no ‘correct’ order in which to write a sum (or product), but experience often suggests which form is preferable in a particular context.

Cancelling common factors A basic way to simplify a fraction is to cancel any (non-zero) *factor* which appears in both the numerator and the denominator. For example, we have

$$\frac{60a^2b}{35a^5c} = \frac{12b}{7a^3c},$$

$$\begin{aligned} a^5 &= a^3 \times a^2, \\ 60 &= 12 \times 5, \\ 35 &= 7 \times 5. \end{aligned}$$

since the factors 5 and a^2 appear in the numerator and in the denominator.

Multiplying out brackets Two sums which are to be multiplied together are each placed within brackets. To find the product, we multiply each term of the first sum by each term of the second, then collect like terms of the resulting sum. For example, we have

$$\begin{aligned} (a-b)(a+b) &= a^2 + ab - ba - b^2 \\ &= a^2 - b^2, \end{aligned}$$

This process is also called ‘expanding brackets’.

$$\begin{aligned} (a+b)^2 &= (a+b)(a+b) \\ &= a^2 + ab + ba + b^2 \\ &= a^2 + 2ab + b^2. \end{aligned}$$

Note that $ab = ba$.

Factorising a sum Factorising an *integer* means expressing it as a product of smaller integers; for example, $18 = 3 \times 6$. Factorising a *sum* means expressing it as a product of simpler sums. This process reverses the effect of multiplying out brackets. For example, as you saw above,

$$\begin{aligned} a^2 - b^2 &\text{ can be factorised as } (a-b)(a+b), \\ a^2 + 2ab + b^2 &\text{ can be factorised as } (a+b)^2. \end{aligned}$$

This first factorisation is called the **difference of two squares**.

Finding a factor is relatively straightforward if all terms of the sum have an identifiable common factor which can be ‘taken out’. For example,

$$x^3 + 2x^2 - 3x = x(x^2 + 2x - 3).$$

Later in this section, we consider the factorisation of expressions such as $x^2 + 2x - 3$.

Combining fractions The rules for combining algebraic fractions are the same as those for combining numerical fractions.

- ◇ To multiply fractions, multiply the numerators and multiply the denominators. For example, we have

$$\left(\frac{a}{2x}\right)\left(\frac{b}{3y}\right) = \frac{ab}{6xy}.$$

- ◇ To divide by a fraction, turn it upside down and multiply. For example, we have

$$\left(\frac{a}{2x}\right) \div \left(\frac{b}{3y}\right) = \left(\frac{a}{2x}\right)\left(\frac{3y}{b}\right) = \frac{3ay}{2bx}.$$

- ◇ To add (or subtract) fractions, first arrange that the fractions have a common denominator, then add (or subtract) the numerators. The simplest way to obtain a common denominator is to multiply the various denominators together. In the following example, a common denominator is $(2x)(3y) = 6xy$:

$$\begin{aligned} \frac{a}{2x} - \frac{b}{3y} &= \left(\frac{a}{2x}\right)\left(\frac{3y}{3y}\right) - \left(\frac{b}{3y}\right)\left(\frac{2x}{2x}\right) \\ &= \frac{3ay}{6xy} - \frac{2bx}{6xy} \\ &= \frac{3ay - 2bx}{6xy}. \end{aligned}$$

Expanding a fraction If the numerator of a fraction is a sum, then the fraction can be written as a sum of simpler fractions. For example, we have

$$\frac{x^2 + 2x + 3}{x^2 + 1} = \frac{x^2}{x^2 + 1} + \frac{2x}{x^2 + 1} + \frac{3}{x^2 + 1}.$$

The expression on the left has been ‘expanded’ to obtain the longer expression on the right.

The next two activities ask you to practise some of these methods of rearranging expressions, and show you how they can be performed in Mathcad.

Activity 4.1 Rearranging algebraic expressions

- (a) Collect like terms in the expression

$$(a - 2b + 3c - 4d) - (-4a + 3b - 2c + d).$$

- (b) Cancel common factors in the fraction

$$\frac{20a^2b}{15ab^2}.$$

- (c) Multiply out each of the following products.

$$(i) (x + 1)(x - 2) \quad (ii) (a - b)^2 \quad (iii) (t + 1)(t + 2)(t + 3)$$

- (d) Factorise each of the following expressions.

$$(i) a^2b - ab^2 \quad (ii) x^2 - 4 \quad (iii) x^2 - 2x + 1$$

Turning a fraction upside down gives the fraction’s **reciprocal**.

Usually, the first step in this rearrangement is not explicitly written down. With practice, it is possible to write down the final step immediately.

- (e) Perform the following addition of fractions, giving the numerator and denominator in factorised form:

$$\frac{4}{3(x+2)} + \frac{2}{x+3}.$$

- (f) Expand the following fraction, simplifying each fraction of the resulting sum:

$$\frac{x^2 + 2x + 1}{x^3}.$$

Solutions are given on page 41.

Switch on your computer and run Mathcad.



Activity 4.2 Rearranging algebraic expressions in Mathcad

Locate in the **Chapter A0** folder the file **121A0-05 Rearranging algebraic expressions**. Open it, save your own copy, and work through the worksheet.

Comment

This activity shows that Mathcad can rearrange algebraic expressions, though it may produce answers in unexpected forms. For the examples in Activity 4.1, there is no reason to use Mathcad, since these can all be done readily by hand. For more complicated examples, such as multiplying out $(x-1)(x-2)(x-3)(x-4)$, Mathcad is useful (if only as a check on your working!).

4.2 Solving equations

An **equation** is any statement in which two expressions are placed on either side of an $=$ sign to indicate that they are equal. Equations may be of several types, as follows.

- ◇ As you have seen, an equation may indicate that two expressions are equivalent. For example, the product $3 \times 6 = 18$ and the factorisation $a^2 - b^2 = (a-b)(a+b)$ are of this type. Such an equation is sometimes called an **identity**.
- ◇ An equation may define a dependent variable in terms of one or more independent variables. An example is the equation $A = \pi r^2$, which defines the value of the variable A . Such an equation is often called a **formula**, and the variable defined is called the **subject** of the formula.
- ◇ An equation may represent a **condition** which is present in a particular problem. For example, suppose that a number x has the property that if we double x and then add 1, the answer obtained is 85. This property can be expressed by the equation $2x + 1 = 85$. Here, the aim is to find the value of x for which the condition is satisfied; such a variable x is called an **unknown**. In an equation with several variables, we often wish to express one unknown variable in terms of the others, as a formula.

It is usually clear from the context which meaning a given $=$ sign has. When reading and writing algebra, it will help if you are conscious of these different meanings.

The symbol $\equiv$ (instead of $=$) is sometimes used for an identity, but not in MST121.

You saw in Activity 3.4 that Mathcad uses the symbol $:=$ to define the value of a variable.

The letters x , y and z are often used for unknown variables, with a , b , c , and so on, for others. There is no 'rule' about this, however!

Two equations are called **equivalent** if one can be obtained from the other by successively changing the equation in the following ways:

- ◇ applying any of the operations $+$, $-$, $\times$, $\div$ in the same way to both sides of the equation (avoiding multiplication or division by 0);
- ◇ replacing any of the expressions in the equation by equivalent expressions;
- ◇ exchanging the left-hand and right-hand sides.

Solving an equation (or equations) means finding all the values of the unknown (or unknowns) for which the equation is satisfied. Such a value is called a **solution** of the equation.

Linear equations

A **linear expression** is a sum, such as $2x + 3y + 1$, consisting of first powers of the variables and a number. A **linear equation** in one (unknown) variable x is an equation of the form

$$ax + b = p,$$

where a , b , x and p are variables, with $a \neq 0$. For example, the equation

$$2x + 1 = 85$$

is linear in x . To solve this equation, we rearrange it as an equivalent equation such that x is the subject – isolated on the left – as follows. First, subtract 1 from both sides to obtain the equivalent equation

$$2x = 85 - 1 = 84,$$

then divide both sides by 2 to give the solution

$$x = \frac{84}{2} = 42.$$

For the general linear equation $ax + b = p$, we can express the unknown variable x in terms of the variables a , b and p in a similar manner; the final result is $x = (p - b)/a$.

A more challenging example is to solve the equation

$$\frac{3}{x+1} = \frac{5}{x+3}. \quad (4.1)$$

This equation makes sense only if $x + 1$ and $x + 3$ are non-zero; that is, if $x \neq -1$ and $x \neq -3$. Assuming these conditions, we can multiply both sides of equation (4.1) by the product $(x + 1)(x + 3)$ to obtain

$$\frac{3(x+1)(x+3)}{x+1} = \frac{5(x+1)(x+3)}{x+3}.$$

By cancelling common factors on both sides, we obtain

$$3(x+3) = 5(x+1).$$

This equation is equivalent to a linear equation, and can be solved by the following simple rearrangements:

$$3x + 9 = 5x + 5 \quad (\text{multiplying out brackets on each side});$$

$$4 = 2x \quad (\text{subtracting 5 and } 3x \text{ from each side});$$

$$x = 2 \quad (\text{dividing each side by 2, and exchanging the two sides}).$$

Thus the solution is $x = 2$. Notice that for this value of x , both $x + 1$ and $x + 3$ are indeed non-zero and, moreover, both sides of equation (4.1) are equal (to 1), as expected.

Such equations are called ‘linear’ because they are closely related to straight lines.

Strictly speaking, these are two equivalent equations; this is a common way of stringing equations together.

The symbol $\neq$ means ‘is not equal to’.

It is good practice to check that your solution satisfies the original equation.

Remark You may be used to a less formal approach to several of these rearrangements, which can be described as

taking an expression to the other side and reversing its role.

For example, you have just seen the following.

- ◇ The equation $\frac{3}{x+1} = \frac{5}{x+3}$ became $3(x+3) = 5(x+1)$.
- ◇ $+5$ on the right became -5 on the left, and $+3x$ on the left became $-3x$ on the right.
- ◇ $\times 2$ on the right became $\div 2$ on the left.

The first of these informal procedures is often called ‘cross-multiplying’. It can be stated in general as

$$\frac{a}{b} = \frac{c}{d} \text{ is equivalent to } ad = bc,$$

provided that $b \neq 0$ and $d \neq 0$.

If you are already confident about this informal approach, then continue to use it. However, if you are at all uncertain about a rearrangement, then it is safest to ‘do the same to both sides’.

Simultaneous linear equations

If more than one equation has to be satisfied by the unknowns in a problem, then these equations are called **simultaneous equations**. For example,

$$\begin{aligned} 2x + 3y &= 5, \\ 5x - 2y &= -16, \end{aligned} \tag{4.2}$$

are two simultaneous linear equations involving the unknowns x and y . The solutions for x and y must satisfy both equations at once.

There are several algebraic methods of solving such simultaneous linear equations, two of which are now reviewed briefly.

Substitution

In this first method, we do the following.

1. Rearrange one of the equations so that one unknown is equal to an expression involving the other.
2. Substitute this expression in the other equation.
3. Solve the resulting linear equation for the other unknown.
4. Substitute this solution into either of the original equations to find the remaining unknown.

These two methods are very similar, but sometimes one is more convenient than the other.

Either equation and either unknown may be chosen.

For example, with equations (4.2) we can work as follows.

1. Rearrange the first equation, $2x + 3y = 5$, to give

$$y = \frac{1}{3}(5 - 2x).$$

2. Substitute this expression for y in the second equation, to give

$$5x - \frac{2}{3}(5 - 2x) = -16; \quad \text{that is, } (5 + \frac{4}{3})x - \frac{10}{3} = -16.$$

3. Solve the above equation, to give $x = -2$.
4. Substitute $x = -2$ in the first equation, $2x + 3y = 5$, to find $y = 3$.

Thus the solutions are $x = -2$ and $y = 3$.

Check:

$$\begin{aligned} 2x + 3y &= 2 \times (-2) + 3 \times 3 = 5, \\ 5x - 2y &= 5 \times (-2) - 2 \times 3 = -16. \end{aligned}$$

Either unknown may be chosen.

Elimination

In this second method, we do the following.

1. Multiply the two equations by numbers chosen so that one of the unknowns has the same coefficient, possibly with the opposite sign, in both equations.
2. Subtract or add the new equations to eliminate that unknown.
3. Solve the resulting linear equation for the other unknown.
4. Substitute this solution into either of the original equations to find the remaining unknown.

For example, with equations (4.2) we can work as follows.

1. Multiply the first equation by 5 and the second by 2, so that x has the same coefficient in each new equation:

$$10x + 15y = 25,$$

$$10x - 4y = -32.$$

2. Subtract the second of *these* equations from the first, to give

$$15y - (-4y) = 25 - (-32); \quad \text{that is,} \quad 19y = 57.$$

3. Solve the above equation, to give $y = 3$.
4. Substitute $y = 3$ in $2x + 3y = 5$, to find $x = -2$.

Thus, as before, the solutions are $x = -2$ and $y = 3$.

For these simultaneous equations, elimination is more convenient.

Quadratic equations

A **quadratic expression** in the variable x is an algebraic expression which may be put in the form

$$ax^2 + bx + c,$$

The case $a = 0$ is excluded because the expression would then be linear (unless $b = 0$ also).

where $a \neq 0$. Correspondingly, a **quadratic equation** in the unknown x is an equation of the form

$$ax^2 + bx + c = p.$$

It is always possible, however, to subtract p from both sides of this equation and hence produce an equivalent quadratic equation which has 0 on the right-hand side. We shall therefore assume that this has been done from the outset, and confine attention to quadratic equations of the form

$$ax^2 + bx + c = 0, \quad \text{where } a \neq 0.$$

For example, one such equation is

$$x^2 + 5x - 14 = 0.$$

Here $a = 1$, $b = 5$ and $c = -14$.

Solving quadratic equations is more complicated than solving linear equations. There are various methods, one of which is *factorisation*.

Factorising a quadratic Some quadratic expressions with integer coefficients can be conveniently expressed as a product of two simple linear factors. For example, we have

$$x^2 + 5x + 6 = (x + 2)(x + 3),$$

$$5x^2 - 3x - 2 = (5x + 2)(x - 1),$$

$$x^2 + 4x + 4 = (x + 2)^2.$$

When a quadratic equation is expressed in factorised form, it is straightforward to solve the equation. For example, the two sides of the equation

$$(5x + 2)(x - 1) = 0$$

are equal when either $5x + 2 = 0$ or $x - 1 = 0$. Therefore this equation has two solutions: $x = -\frac{2}{5}$ and $x = 1$.

Factorisation is a very effective way to solve a quadratic equation, *when it can be done*. But deciding whether a convenient factorisation is available can be tricky, and requires a level of skill which develops only with practice. Consider the special case where $a = 1$, and b and c are integers. We then seek linear factors of $x^2 + bx + c$ of the form $x + \alpha$ and $x + \beta$. Now

$$(x + \alpha)(x + \beta) = x^2 + (\alpha + \beta)x + \alpha\beta.$$

Comparing this right-hand side with $x^2 + bx + c$, we see that α and β must be chosen so that

$$\alpha + \beta = b \quad \text{and} \quad \alpha\beta = c.$$

Therefore, for a simple factorisation, we need to consider the pairs of integers α and β whose product is c , and check whether any of these have sum b . For example, to solve

$$x^2 + 5x - 14 = 0$$

by factorisation, we need to find two integers whose product is $c = -14$ and whose sum is $b = 5$. After considering the various possibilities, the values 7 and -2 are found to satisfy both of these conditions, so we obtain the factorisation

$$x^2 + 5x - 14 = (x + 7)(x - 2).$$

Thus the solutions of the equation $x^2 + 5x - 14 = 0$ are $x = -7$ and $x = 2$.

Since such a convenient factorisation is not always available, we need another approach.

Solving quadratic equations by the formula Fortunately, there is a formula which enables us to find the solutions of *any* quadratic equation. This states that the solutions of the equation

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (4.3)$$

It is good practice to check solutions, but substituting solutions in the original equation(s) can be a time-consuming task!

The symbols α and β are the Greek letters alpha and beta. All the Greek letters are listed in *Guide to Preparation* and also in the *Handbook*.

The possible pairs with product -14 are:

$$\begin{array}{ll} 14, -1; & 7, -2; \\ 1, -14; & 2, -7. \end{array}$$

Of these, only the pair 7, -2 has sum 5:

$$7 + (-2) = 5.$$

This formula is derived later in MST121.

Here the symbol $\pm$ denotes two alternative solutions, one with the plus sign and the other with the minus sign.

For example, to solve the equation

$$x^2 - x - 1 = 0,$$

we use equation (4.3) with $a = 1$, $b = -1$ and $c = -1$, to obtain

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-1)}}{2 \times 1} = \frac{1}{2}(1 \pm \sqrt{5}).$$

The decimal values of these two solutions are

$$\frac{1}{2}(1 + \sqrt{5}) = 1.618 \text{ (to 3 d.p.)}, \quad \frac{1}{2}(1 - \sqrt{5}) = -0.618 \text{ (to 3 d.p.)}.$$

The fact that these solutions are not integers confirms that there was no simple factorisation of the quadratic expression $x^2 - x - 1$.

A ‘real solution’ is a solution which is a real number.

Equation (4.3) indicates that some quadratic equations have real solutions, but others do not. For real solutions, the quantity beneath the square root sign must be positive or 0. Hence there are:

two real solutions if $b^2 - 4ac > 0$;
 one real solution if $b^2 - 4ac = 0$;
 no real solutions if $b^2 - 4ac < 0$.

For example, the equation $x^2 + 4x + 4 = 0$ has one real solution ($x = -2$), because in this case

$$b^2 - 4ac = 4^2 - 4 \times 1 \times 4 = 0.$$

On the other hand, the equation $x^2 + 2x + 3 = 0$ has no real solutions, because in this case

$$b^2 - 4ac = 2^2 - 4 \times 1 \times 3 = -8.$$

Remark Solutions of an equation are often called *roots* of the equation. In the case of a quadratic equation with $b^2 - 4ac = 0$, the one solution is called a *repeated solution* or *repeated root*.

The next activity asks you to practise solving linear and quadratic equations.

Activity 4.3 Solving equations

(a) Solve the equation

$$2x - 3 = 7.$$

(b) Solve the following equation to find x in terms of a and b , assuming that $a \neq b$:

$$a(x + a) = b(x + b).$$

(c) Solve the pair of simultaneous equations

$$\begin{aligned} 2x + 6y &= -11, \\ 4x - 3y &= -2. \end{aligned}$$

(d) Solve each of the following quadratic equations.

(i) $x^2 + 2x - 3 = 0$

(ii) $2x^2 + 6x - 5 = 0$

(iii) $6x^2 - 13x + 6 = 0$

Solutions are given on page 42.

Later in MST121 you will see how Mathcad is used to solve equations. In particular, you will learn how to use the symbolic keyword ‘solve’ to solve quadratic equations.

Summary of Section 4

This section recalled basic rearrangements of algebraic expressions and equations, and showed you how some of these rearrangements can be carried out with Mathcad. In particular, you saw:

- ◇ the notation for algebraic expressions and equations;
- ◇ the rules for rearranging algebraic expressions in order to arrive at equivalent expressions;
- ◇ the rules for rearranging equations in order to arrive at equivalent equations;
- ◇ how to solve simultaneous linear equations by the methods of substitution and elimination;
- ◇ how to solve a quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, by the method of factorisation and by using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};$$

- ◇ the use of Mathcad's symbolic keywords 'simplify', 'expand' and 'factor'.

Exercises for Section 4

Exercise 4.1

Use hand calculation, your calculator or Mathcad, as appropriate, to do each of the following.

To use Mathcad, you will need to create a new (Normal) worksheet.

- (a) Collect like terms in the expression

$$3x - 2y + 4z + 6x + 2y - 3z - 2x - 3y + z.$$

- (b) Cancel common factors in the fraction

$$\frac{8a^2bc^2}{6a^2c}.$$

- (c) Multiply out each of the following products.

$$(i) (x-1)(y+2) \quad (ii) (x-1)^2(y+2)^2$$

- (d) Factorise the following expressions.

$$(i) 4u^2 - 9 \quad (ii) 5ax^2 - 10bx \quad (iii) x^2 - 4x + 4 \quad (iv) a^3 - b^3$$

- (e) Perform the following addition of fractions:

$$\frac{1}{x+2} - \frac{2}{2x+5}.$$

- (f) Expand the following fraction, simplifying each fraction of the resulting sum:

$$\frac{a^4 + 1}{a^2}.$$

Exercise 4.2

Use hand calculation or your calculator, as appropriate, to do each of the following.

- (a) Solve the following equation to find x in terms of a and b , assuming that $a \neq b$:

$$(x + a)b = ax.$$

- (b) Solve the pair of simultaneous equations

$$3a + 5b = 21,$$

$$2a + 3b = 13.$$

- (c) Solve each of the following equations.

(i) $x^2 + x - 12 = 0$ (ii) $1.1x^2 + 0.99x - 12.01 = 0$

Summary of Chapter A0

After studying a chapter, it is suggested that you spend some time reviewing the material in order to consolidate your knowledge. To help you in this, each chapter's key points are listed in a summary like the one below.

In this chapter, you saw how to install Mathcad, the Mathcad files and MST121 software on your computer, before taking some first steps in using Mathcad. You recalled basic properties of numbers and arithmetic, and met several types of numbers. You also recalled basic rearrangements of algebraic expressions and equations.

Learning outcomes

You have been working towards the following learning outcomes.

Terms to know and use

Natural numbers, integers, real numbers, rational and irrational numbers, finite, infinite and recurring decimals, real line, power, variable, independent and dependent variables, algebraic expression, term, sum, product, factor, coefficient, numerator, denominator, fraction, quotient, identity, formula, subject, condition, unknown, linear expression, linear equation, simultaneous equations, quadratic expression, quadratic equation, factorisation.

Symbols and notation to know and use

$\mathbb{N}$ (natural numbers), $\mathbb{Z}$ (integers), $\mathbb{Q}$ (rational numbers), $\mathbb{R}$ (real numbers);
dot notation for recurring decimals;
notation for powers, including roots;
scientific notation.

Mathematical skills

- ◇ Evaluate arithmetic expressions, by hand or calculator.
- ◇ Use variables.
- ◇ Rearrange algebraic expressions.
- ◇ Solve simultaneous linear equations and quadratic equations.
- ◇ Determine the number of real solutions of a quadratic equation.

Mathcad skills

- ◇ Access Mathcad and the Mathcad files (having loaded the software onto your hard disk).
- ◇ Create, navigate around, save, print and close a Mathcad worksheet.
- ◇ Use Mathcad for basic calculations and basic algebra.

Ideas to be aware of

- ◇ Whether to use hand calculation, your calculator or Mathcad may depend on the type of calculation to be performed.
- ◇ The equals sign $=$ has several different meanings.

Appendix: Numerical form of answers

In Section 3, you were reminded that there are several ways of representing real numbers, including decimals, scientific notation, fractions and roots, and that decimal representations may be rounded either to a given number of decimal places or to a given number of significant figures. One consequence of this is that you will have to decide how to present numerical answers to questions. The following guidelines are offered to help you with this.

- ◇ Always check to see whether a question (especially one in an assignment or examination) asks for numerical answers in a particular form.
- ◇ If no particular form of answer is specified or suggested by the context, then decimal answers will usually be rounded to three decimal places or three significant figures.
- ◇ In questions where the calculations involve only simple fractions, the answers should be given in this form.
- ◇ Even if you are going to round an *answer*, try to perform all the calculations to the full accuracy of whatever method (hand, calculator, Mathcad) you are using. Round answers only at the final stage of presenting them; in particular, do *not* use rounded answers to earlier parts of a question in subsequent calculations (unless instructed to do so).

For example, in the solutions for this chapter, numbers obtained from a calculator are recorded to eight significant figures in the first instance.

Solutions to Activities

In these solutions, all decimals obtained by a calculator are given at first to eight significant figures.

Solution 3.1

- (a) Since $625 = 25^2 = 5^4$, we have

$$625^{3/4} = \left(\sqrt[4]{625}\right)^3 = 5^3 = 125.$$

- (b) By calculator,

$$\frac{78}{98} = 0.79591837 = 0.795918 \text{ (to 6 d.p.)}.$$

- (c) By calculator,

$$2^{4/3} = 2.5198421 = 2.520 \text{ (to 3 d.p.)}.$$

- (d) By calculator,

$$\begin{aligned} 2^{100} &= 1.2676506 \times 10^{30} \\ &= 1.27 \times 10^{30} \text{ (to 3 s.f.)} \end{aligned}$$

Solution 3.3

- (a) For $r = 2$,

$$\begin{aligned} C &= 2\pi r \\ &= 2 \times \pi \times 2 \\ &= 12.566371 = 12.6 \text{ (to 3 s.f.)} \end{aligned}$$

For $r = 5$,

$$\begin{aligned} C &= 2\pi r \\ &= 2 \times \pi \times 5 \\ &= 31.415927 = 31.4 \text{ (to 3 s.f.)} \end{aligned}$$

- (b) For $r = 1.5$,

$$\begin{aligned} A &= \pi r^2 \\ &= \pi \times (1.5)^2 \\ &= 7.0685835 = 7.07 \text{ (to 3 s.f.)} \end{aligned}$$

For $r = 3.7$,

$$\begin{aligned} A &= \pi r^2 \\ &= \pi \times (3.7)^2 \\ &= 43.008403 = 43.0 \text{ (to 3 s.f.)} \end{aligned}$$

- (c) For $r = 1.5$ and $h = 8$,

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi \times (1.5)^2 \times 8 \\ &= 56.548668 = 56.5 \text{ (to 3 s.f.)} \end{aligned}$$

- (d) For $a = 1$ and $b = 2$,

$$\begin{aligned} c &= \sqrt{a^2 + b^2} \\ &= \sqrt{1 + 4} \\ &= 2.2360680 = 2.24 \text{ (to 3 s.f.)} \end{aligned}$$

For $a = 3$ and $b = 4$,

$$\begin{aligned} c &= \sqrt{a^2 + b^2} \\ &= \sqrt{9 + 16} \\ &= 5. \end{aligned}$$

Solution 4.1

$$\begin{aligned} \text{(a)} \quad (a - 2b + 3c - 4d) - (-4a + 3b - 2c + d) \\ = 5a - 5b + 5c - 5d \end{aligned}$$

$$\text{(b)} \quad \frac{20a^2b}{15ab^2} = \frac{4a}{3b}$$

$$\text{(c)} \quad \begin{aligned} \text{(i)} \quad (x + 1)(x - 2) &= x^2 - 2x + x - 2 \\ &= x^2 - x - 2 \end{aligned}$$

$$\text{(ii)} \quad (a - b)^2 = a^2 - ab - ba + b^2 = a^2 - 2ab + b^2$$

(iii) Multiplying out the first two factors first gives

$$\begin{aligned} (t + 1)(t + 2)(t + 3) \\ = (t^2 + 3t + 2)(t + 3) \\ = t^3 + 3t^2 + 3t^2 + 9t + 2t + 6 \\ = t^3 + 6t^2 + 11t + 6. \end{aligned}$$

Alternatively, the last two factors could be multiplied first.

- (d) (i) The term ab is a common factor, so

$$a^2b - ab^2 = ab(a - b).$$

(ii) The expression $x^2 - 4$ is a difference of two squares, so

$$x^2 - 4 = (x - 2)(x + 2).$$

(iii) The expression $x^2 - 2x + 1$ has the same form as $a^2 - 2ab + b^2$, obtained in part (c)(ii), with x instead of a and 1 instead of b , so

$$x^2 - 2x + 1 = (x - 1)^2.$$

- (e) A common denominator is $3(x + 2)(x + 3)$, so

$$\begin{aligned} \frac{4}{3(x + 2)} + \frac{2}{x + 3} &= \frac{4(x + 3) + 2 \times 3(x + 2)}{3(x + 2)(x + 3)} \\ &= \frac{10x + 24}{3(x + 2)(x + 3)} \\ &= \frac{2(5x + 12)}{3(x + 2)(x + 3)}. \end{aligned}$$

- (f) Dividing each term of the numerator by the denominator, and simplifying, gives

$$\frac{x^2 + 2x + 1}{x^3} = \frac{1}{x} + \frac{2}{x^2} + \frac{1}{x^3}.$$

Solution 4.3

(a) Starting from

$$2x - 3 = 7,$$

use the following rearrangements:

$$2x = 10 \quad (\text{adding } 3 \text{ to both sides});$$

$$x = 5 \quad (\text{dividing both sides by } 2).$$

(b) Starting from

$$a(x + a) = b(x + b),$$

use the following rearrangements:

$$ax + a^2 = bx + b^2 \quad (\text{multiplying out brackets});$$

$$(a - b)x = b^2 - a^2 \quad (\text{collecting like terms});$$

$$(a - b)x = (b - a)(b + a) \quad (\text{difference of two squares}).$$

The assumption $a \neq b$ means that $a - b \neq 0$, so we can divide both sides by $a - b$, to give

$$\begin{aligned} x &= \frac{(b - a)(b + a)}{a - b} \\ &= -\frac{(a - b)(b + a)}{a - b} \\ &= -(b + a) = -a - b. \end{aligned}$$

(c) To solve

$$2x + 6y = -11, \quad (1)$$

$$4x - 3y = -2, \quad (2)$$

we first multiply equation (1) by 2 so that the coefficients of x are the same in both equations:

$$4x + 12y = -22, \quad (3)$$

$$4x - 3y = -2. \quad (2)$$

Then subtract equation (2) from equation (3), to give

$$15y = -20,$$

$$\text{so } y = -\frac{4}{3}.$$

On substituting this value for y into equation (1), we obtain

$$2x = -11 - 6 \times \left(-\frac{4}{3}\right) = -3,$$

$$\text{so } x = -\frac{3}{2}.$$

Thus the solutions are $x = -\frac{3}{2}$ and $y = -\frac{4}{3}$.

(d) (i) To solve

$$x^2 + 2x - 3 = 0,$$

note the factorisation

$$x^2 + 2x - 3 = (x - 1)(x + 3),$$

which gives the solutions $x = 1$ and $x = -3$.

Alternatively, use the formula, which gives

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times (-3)}}{2 \times 1} \\ &= \frac{1}{2}(-2 \pm \sqrt{16}) \\ &= \frac{1}{2}(-2 \pm 4) \\ &= -1 \pm 2, \end{aligned}$$

so $x = 1$ and $x = -3$ again.

(ii) To solve

$$2x^2 + 6x - 5 = 0$$

(assuming that no convenient factorisation occurs to you), use the formula:

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{6^2 - 4 \times 2 \times (-5)}}{2 \times 2} \\ &= \frac{1}{4}(-6 \pm \sqrt{76}) \\ &= \frac{1}{4}(-6 \pm \sqrt{4 \times 19}) \\ &= \frac{1}{4}(-6 \pm 2\sqrt{19}) \\ &= \frac{1}{2}(-3 \pm \sqrt{19}). \end{aligned}$$

In decimals, the solutions are

$$x = 0.679\,449\,47 = 0.679 \text{ (to 3 d.p.) and}$$

$$x = -3.679\,449\,5 = -3.679 \text{ (to 3 d.p.).}$$

(In the above calculation, the rule

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

has been used, with $ab = 76$, $a = 4$ and $b = 19$.)

(iii) To solve

$$6x^2 - 13x + 6 = 0$$

(assuming that no convenient factorisation occurs to you), use the formula:

$$\begin{aligned} x &= \frac{-(-13) \pm \sqrt{(-13)^2 - 4 \times 6 \times 6}}{2 \times 6} \\ &= \frac{1}{12}(13 \pm \sqrt{25}) \\ &= \frac{1}{12}(13 \pm 5), \end{aligned}$$

$$\text{giving } x = \frac{18}{12} = \frac{3}{2} \text{ and } x = \frac{8}{12} = \frac{2}{3}.$$

(The fact that these solutions are fractions indicates that there is a simple factorisation, namely

$$6x^2 - 13x + 6 = (3x - 2)(2x - 3),$$

but it might have taken some time to find this!)

Solutions to Exercises

Solution 3.1

- (a) (i) Since $49 = 7^2$, we have

$$49^{3/2} = \left(\sqrt{49}\right)^3 = 7^3 = 343.$$

- (ii) By calculator,

$$50^{3/2} = 353.553\,39 = 354 \text{ (to 3 s.f.)}.$$

- (iii) Since $26 = 2 \times 13$ and $65 = 5 \times 13$, we can cancel the factor 13 in the numerator and denominator, to give

$$\frac{26}{65} = \frac{2 \times 13}{5 \times 13} = \frac{2}{5} = 0.4.$$

- (iv) By calculator,

$$\frac{27}{65} = 0.415\,384\,62 = 0.415 \text{ (to 3 s.f.)}.$$

- (v) By calculator,

$$\begin{aligned} 3^{-10} &= 0.000\,016\,935\,088 \\ &= 1.69 \times 10^{-5} \text{ (to 3 s.f.)}. \end{aligned}$$

Solution 3.2

- (a) (i) For $r = 2.5$,

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3} \times \pi \times (2.5)^3 \\ &= 65.449\,847 = 65.4 \text{ (to 3 s.f.)}. \end{aligned}$$

For $r = 4.7$,

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3} \times \pi \times (4.7)^3 \\ &= 434.892\,77 = 435 \text{ (to 3 s.f.)}. \end{aligned}$$

- (ii) For $a = 2$, $b = 3$ and $c = 4$,

$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(2 + 3 + 4) = 4.5$$

and

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{4.5 \times 2.5 \times 1.5 \times 0.5} \\ &= 2.904\,737\,5 = 2.90 \text{ (to 3 s.f.)}. \end{aligned}$$

Solution 4.1

- (a) The expression

$$3x - 2y + 4z + 6x + 2y - 3z - 2x - 3y + z$$

has terms in x , y and z , which can be collected to give

$$7x - 3y + 2z.$$

- (b) The numerator and denominator have common factors 2, a^2 and c , so

$$\frac{8a^2bc^2}{6a^2c} = \frac{4bc}{3} = \frac{4}{3}bc.$$

- (c) (i) We have

$$(x-1)(y+2) = xy + 2x - y - 2.$$

- (ii) To expand

$$(x-1)^2(y+2)^2 = (x^2 - 2x + 1)(y^2 + 4y + 4)$$

involves a more complicated calculation, and there is a good chance of making a mistake. This seems a good time to use Mathcad's symbolic keyword 'expand', which gives the answer

$$\begin{aligned} &x^2y^2 + 4x^2y + 4x^2 - 2xy^2 \\ &- 8xy - 8x + y^2 + 4y + 4. \end{aligned}$$

- (d) (i) The expression $4u^2 - 9$ is a difference of two squares:

$$4u^2 - 9 = (2u)^2 - 3^2.$$

Therefore

$$4u^2 - 9 = (2u - 3)(2u + 3).$$

- (ii) The terms of $5ax^2 - 10bx$ have common factors 5 and x , so

$$5ax^2 - 10bx = 5x(ax - 2b).$$

- (iii) The expression $x^2 - 4x + 4$ is of the form $a^2 - 2ab + b^2$, with x instead of a and 2 instead of b . Thus

$$x^2 - 4x + 4 = (x - 2)^2.$$

- (iv) The expression $a^3 - b^3$ seems difficult to factorise (unless you have seen it factorised before). Mathcad's symbolic keyword 'factor' gives

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2),$$

which you can check by multiplying out the brackets on the right.

(Your Mathcad answer may appear in a form slightly different to this; for example, with the factor $-(b-a)$ in place of $(a-b)$, or with the term ba in place of ab .)

- (e) A common denominator is $(x+2)(2x+5)$, so

$$\begin{aligned} \frac{1}{x+2} - \frac{2}{2x+5} &= \frac{(2x+5) - 2(x+2)}{(x+2)(2x+5)} \\ &= \frac{1}{(x+2)(2x+5)}. \end{aligned}$$

- (f) We have

$$\frac{a^4 + 1}{a^2} = \frac{a^4}{a^2} + \frac{1}{a^2} = a^2 + \frac{1}{a^2}.$$

Solution 4.2

(a) Starting from

$$(x + a)b = ax,$$

use the following rearrangements:

$$bx + ab = ax \quad (\text{multiplying out brackets});$$

$$(b - a)x = -ab \quad (\text{collecting like terms}).$$

Since $a \neq b$, we can divide both sides by $b - a$, to give

$$x = \frac{-ab}{b - a} = \frac{ab}{a - b}.$$

(b) To solve

$$3a + 5b = 21, \quad (1)$$

$$2a + 3b = 13, \quad (2)$$

we multiply equation (1) by 2 and equation (2) by 3, so that the coefficients of a are the same:

$$6a + 10b = 42, \quad (3)$$

$$6a + 9b = 39. \quad (4)$$

Then subtract equation (4) from equation (3), to give $b = 3$.

On substituting this value for b into equation (1), we obtain

$$3a = 21 - 5 \times 3 = 6,$$

so $a = 2$.

Thus the solutions are $a = 2$ and $b = 3$.

(c) (i) To solve

$$x^2 + x - 12 = 0,$$

note the factorisation

$$x^2 + x - 12 = (x + 4)(x - 3),$$

which gives the solutions $x = -4$ and $x = 3$.

Alternatively, use the formula, which gives

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times (-12)}}{2 \times 1} \\ &= \frac{1}{2}(-1 \pm \sqrt{49}) \\ &= \frac{1}{2}(-1 \pm 7), \end{aligned}$$

so $x = 3$ and $x = -4$ again.

(ii) To solve

$$1.1x^2 + 0.99x - 12.01 = 0,$$

we use the formula (a convenient factorisation seems most unlikely!):

$$\begin{aligned} x &= \frac{-0.99 \pm \sqrt{0.99^2 - 4 \times 1.1 \times (-12.01)}}{2 \times 1.1} \\ &= \frac{-0.99 \pm \sqrt{0.9801 + 52.844}}{2.2} \\ &= \frac{-0.99 \pm \sqrt{53.8241}}{2.2} \\ &= \frac{-0.99 \pm 7.3364910}{2.2}. \end{aligned}$$

Hence we have

$$\begin{aligned} x &= \frac{6.3464910}{2.2} = 2.8847686 \\ &= 2.88 \text{ (to 3 s.f.)} \end{aligned}$$

and

$$\begin{aligned} x &= \frac{-8.3264910}{2.2} = -3.7847686 \\ &= -3.78 \text{ (to 3 s.f.)}. \end{aligned}$$

(Notice that the coefficients in the two quadratic equations in parts (i) and (ii) are quite close, and the solutions are similarly close; this provides one possible check on the working.)

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